The Forward Exchange Rate and the Interest Rate within a Production Economy

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Abstract

This paper considers some equilibrium characteristics of the forward foreign exchange rate and the interest rate within a production economy. In order to derive the optimal solutions, a two-stage decision process is assumed; the first is the production decision, and the second is the portfolio decision. With some additional assumptions, both the forward exchange rate and the interest rate are shown to depend on the underlying stochastic parameter in the production function. Unlike in an exchange economy, the unbiasedness hypothesis of the forward exchange rate fails because of random technological shocks in production. (JEL Classification: F31, G11, E21)
I. Introduction

Recent economic models, including monetary theory, are characterized by two distinct features. One is that the model assumes dynamic elements explicitly, and the other is that it also considers inherently stochastic elements.

Although it has long been recognized that both these elements are important in economic analyses, it is relatively recently that their economic implications have been examined simultaneously within a single unified model. A notable example is the so-called “equity premium puzzle” (Mehra and Prescott [1985]), and its related and descendent analyses (for a recent survey of the literature, see Kudoh [1991]). Those problems have been examined by the so-called “recursive method”, which is expounded in, e.g., Stokey, Lucas and Prescott [1989] (for a recent survey of the literature, see Judd [1991]). The method makes it possible to solve a wide range of stochastic dynamic optimization problems and to generate the agent’s optimal decision rules that characterize the description of the equilibrium.

With the help of the aforementioned methods this article conducts an equilibrium analysis within a single model of production economy, without constructing different models under the traditional economics of uncertainty. To be more specific, this article examines, within a simple and single but unified model, two separate but closely related issues that have been traditionally analyzed in two different models, i.e., the optimal consumption-saving decision as a solution of the constrained utility maximization, and the optimal portfolio choice as a solution of a simple decision problem.1

To solve the problems, I focused on the unbiasedness hypothesis of the forward foreign exchange rate and on the equilibrium rate of interest within a simple real business cycle model which is nothing but a growth model with random technological shocks (for recent surveys of the literature, see Plosser [1989], Mankiw [1989], and Stadler [1994]).2

These topics have traditionally been examined within a simple two-period

1. For a textbookish exposition of a stage analysis of saving decision under uncertainty, see McKenna [1986].
2. See also Mankiw [1990] and Blanchard and Fischer [1989] for implications of real business cycles in macroeconomics.
model of pure exchange (i.e., no production) by, e.g., Frenkel and Razin [1980]. However, my model considers production explicitly, and thus a similar but slightly different model than those of Donaldson and Mehra [1983] and Mehra [1984] is constructed. In fact, Frenkel and Razin [1980] completely neglected production uncertainty and only concentrated on price uncertainty in an open economy. One of the motivations of the present study lies in this “incompleteness” of previous studies of price uncertainty. Kemp [1976] clearly pointed out that taking price uncertainty as given (as in Frenkel and Razin [1980]) is analytically incomplete “without relating it to the underlying randomness of preferences, technology or factor endowments, . . .” (p. 264).

The unbiasedness of the forward exchange rate has been one of the necessary conditions to preclude speculative behavior in recent literature of the full hedge theorem for the exchange rate risk within a portfolio decision model (e.g., Lehrbass [1994]). Since the problems are slightly complicated, I simplified the model, without loss of generality, by parametrization, and derived a condition for the arbitrage-free forward exchange rate, showing the well-know fact that, in a world inhabited by risk averse individuals, unbiasedness and absence of arbitrage are mutually exclusive. Furthermore, I examined several notable characteristics of the equilibrium forward exchange rate and the equilibrium rate of interest within my assumed model.

The remainder of the article is organized as follows. Section II presents a simple real business cycle model to examine the implications of the optimal consumption decision of an economy consisting of infinitely lived representative agents. Section III considers the portfolio decision as the decision of optimal saving that lies behind the optimal consumption decision. My specific focal points rest on the relationship between the forward exchange rate and the expected future spot exchange rate in a production economy, and the detailed examination of its characteristics and its relationship with the rate of interest. Section IV concludes the paper.

II. The Model

In order to examine some characteristics of an important hypothesis of unbiasedness with respect to the forward foreign exchange rate, and its
relationship with the rate of interest, I constructed and utilized a simple but standard stochastic growth model or a real business cycle model (e.g., Donaldson and Mehra [1983]; Mehra [1984]). The model is described as follows.

Assumption 1: Small open economy
The country under consideration is assumed to be small relative to the rest of the world, and have a negligible influence on the rest-of-the-world variables. This means that these variables can be taken as exogenous in the model.

Assumption 2: Preference
(a) The representative agent’s preference is given by the discounted sum of the time-separable (but state-inseparable) and instantaneous utility (or “felicity”) function:

\[ U(\cdot) = \mathbb{E}\left[ \sum_{t=0}^{\infty} \beta^t u(C_t) \right] \]  

(1)

(b) \( u(\cdot) \) is non-negative and a concave increasing function of real consumption, \( C_t \).

(c) \( \beta \) is a parameter representing the subjective discount rate, or the rate of time preference, which is assumed to be strictly positive.

\( \mathbb{E} \) is an operator representing expectations. For the sake of simplicity and ease of calculation, the planning horizon is made infinite. Also for simplicity’s sake, the felicity function is parametrized as follows:

Assumption 2': Felicity
(b') \( u(C_t) = \ln C_t \)  

(2)

Assumption 2' implies that \( u(\cdot) \) is a function with constant elasticity of substitution, or an iso-elastic function. (2) is also known to belong to a class of CRRA (the constant relative risk aversion) utility function. The assumption of a logarithmic utility function has frequently been adopted for reasons of analytical traceability; see, for example, Blanchard and Fischer [1989], Mehra [1984], and Salyer [1994].

The utility function given by equation (1) is maximized under the follow-
ing budget constraint:
\[ Y_t + R_{t-1}B_{t-1} + S_tR_{t-1}B^*_t + (S_t - F_{t-1})A_{t-1} = C_t + K_{t+1} + (B_t + S_tB^*_t) \] (3)

where:
- \( Y_t \) = the real output of the home country in period \( t \).
- \( B_t(B^*_t) \) = the one-period bond issued by the home (foreign) government in period \( t \).
- \( R_t(R^*_t) \) = the rate of return of the home (foreign) bond in terms of the home (foreign) currency (i.e., one plus home (foreign) rate of interest \( (r_t(r^*_t)) \)).
- \( S_t \) = the spot exchange rate in period \( t \) (expressed as units of home currency per unit of the foreign currency)
- \( F_t \) = the one-period forward foreign exchange rate in period \( t \) (defined similar to \( S_t \)).
- \( A_t \) = the one-period forward foreign exchange contract in period \( t \), to be delivered in period \( t+1 \).
- \( C_t \) = the real consumption in period \( t \), and
- \( K_t \) = the real capital stock in period \( t \).

The left-hand side of (3) represents the real resources that are disposable in period \( t \), while the right-hand side represents the sum of real expenditures in period \( t \). The representative agent makes a forward contract \( A_t \) in period \( t \), which is to be delivered in period \( t+1 \), at the forward price \( F_t \). However, since it is nothing but a contract, it does not appear in the right-hand side in period \( t \), because it will not be delivered until period \( t+1 \).

Assumption 3: No margin requirement
It is assumed in (3) that forward contracts do not entail a margin requirement, so the interest cost is zero.

Assumption 4: Production
(a) The production function
\[ Y_t = Z_tK_t^\alpha \quad 1 > \alpha > 0 \] (4)
i.e., \( Y \) is produced by a concave production function with respect to \( K \).
(b) The logarithm of \( Z_t \) is assumed to follow a first-order autoregressive scheme
\[ \ln Z_t = \rho \ln Z_{t-1} + W_t \]  
\hspace{1cm} (5-1) 

where \( |\rho| < 1 \). \( W_t \) is an i.i.d. random shock, which obeys:

\[ W_t \sim N(0, \sigma^2_w) \]  
\hspace{1cm} (5-2) 

Thus, \( \ln Z_t \) obeys a log-normal distribution with \( E[\ln Z_t] = 0 \) and \( E[(\ln Z_t)^2] = \sigma^2_w/(1 - \rho^2) \).

(c) \( K_t \) depreciates at a rate of 100% in each period.

**Assumption 5: The timing of optimization**

Considering the simple and clear fact that production takes time, the timing of optimization by the representative agent in each period is as follows:

(a) At the first stage the optimal \( C_t \) and investment, and therefore the optimal \( K_t \) (hence output \( Y_t \)) are chosen under the constraints (3) and (4).

(b) At the second stage the optimal portfolio decision \( (B, B^*, A) \) is made.

In other words, in the first stage after actual real output is observed, the representative agent allocates output between optimal current consumption and the next period’s capital. In the second stage, before uncertainty about future exchange rates is resolved, he or she allocates the remaining realized real resource optimally between real assets to transfer it to the next period.\(^3\)

**Assumption 6: Ruling out speculative behavior**

At the first stage of each period the representative agent makes the optimal choice, expecting that the unbiasedness hypothesis holds strictly. In other words, he or she expects \( E[S_{t+i}] = F_{t+i} \) for all positive integer \( i \). Under the assumption the constraint (3) can be rewritten as follows:

\[ Y_t = C_t + K_{t+1} + [(B_{t+1} + S_t B^*_t) - (R_{t-1} B_{t-1} + S_t R^*_t B^*_{t-1})] \]

\(^3\) A similar, but slightly different model of a two-stage decision process under uncertainty is employed by, e.g., Barari and Lapan [1993]. The present analysis is actually motivated partly by an intrinsic criticism first clearly expressed by Kemp [1976] on price uncertainty (see the quotation in the introduction section). Turnovsky [1976] also made the point clear, stating that “... one wishes to... consider the ultimate random disturbances, such as fluctuations in tastes and technological conditions, which presumably are what the random movements in price must be reflecting” (p. 134).
\[ Q_t = [(B_t + S_t B_t^*) - (R_{t-1} B_{t-1} + S_t R_{t-1} B_{t-1}^*)] \]  

\[ Q_t = \text{the net demand for the real bonds in period } t \text{, and can be either positive, negative, or zero.} \]

Regarding \((K_{t+1} + Q_t)\) as a slack variable for the Kuhn-Tucker conditions, maximization of \((1)\) with respect to \(C_t\) under the constraints of \((4)\) and \((6)\) yields the following optimal conditions for the first stage:

\[ C_t = \beta C_{t-1} - Q_t = (1 - \alpha \beta) Z_t K_t^\alpha - Q_t \]  
\[ K_{t+1} = \alpha Y_{t+1} = \alpha \beta Z_t K_t^\beta \]  

This last solution represents the “law of motion” for the capital stock (Mehra [1984], p.277). Substituting these optimal solutions into \((6)\) and rearranging yield the following expressions:

\[ Y_t = \beta^3 Y_0 + (1 - \alpha \beta) \beta^1 Q_0 \]  
\[ Q_t = \beta^1 Q_0 \]  

\(Y_0\) and \(Q_0\) are the initial values of \(Y_t\) and \(Q_t\), respectively, and are assumed to be positive. The first term of the right-hand side of \((7-3)\) is the solution of the homogeneous part, while the second term is the particular solution of the non-homogeneous part, and together they constitute the general solution.

In stating the maximization problem I have not imposed the constraint that the net demand for the real bonds, which is given by \(Q_t\) at time \(t\), be non-negative. Thus, in order to avoid a trivial Ponzi-like solution, the following assumption is imposed:

**Assumption 7:** The No-Ponzi-Game condition

\[ \lim_{t \to \infty} Q_t \exp(-rt) \geq 0 \]  

This condition requires that the net real borrowing should not increase asymptotically faster than the interest rate.

From \((7-3)\) and \((7-4)\) defining \(\gamma = Q_0 / [Y_0 + (1 + \alpha \beta) Q_0]\) (hence \(1 > \gamma > 0\)) yields a simple expression \(Q_t = \gamma Y_t\), which upon substitution into \((7-1)\), yields the following optimal consumption function:
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\[ C_t = (1 - \alpha - \gamma)Y_t \]  

(7-1')

1 - \alpha - \gamma is the APC and the MPC, and it is assumed to be a positive fraction.

Substituting the optimal solution (7-2) into the production function (4) successively, and taking logarithms yield the following condition under the optimality:

\[
\log Y_t = \sum_{i=1}^{t} i^{-1} \log Z_{t+1-i} + \log \sum_{i=1}^{t} i + t \log Y_0
\]

(8-1)

Defining the unconditional mean of \( Y_t \) by the limit of the expectation of (8-1), assumption 4 yields:

\[
E[\log Y_t] = \lim_{t \to \infty} E[\log Y_t] = (\log )/(1 - )
\]

(8-2)

Thus, under the conditions of optimality the optimal real output will disperse around the constant mean value (8-2).

III. The Optimal Portfolio in a Production Economy

Once the optimal decisions of consumption \( C_t \) and investment \( K_t \) (and hence output \( Y_t \)) at the first stage in each planning period are made as summarized in (7-1) and (7-2), as assumed in assumption 5, the remaining problem to be solved at the second stage is the optimal portfolio decision. This decision is made formally by maximizing (1) with respect to \{B_t, B^*_t, A_t\} under the constraint (3). The first-order conditions of the optimization are:

\[
\beta E[C_t/C_{t+1}] = 1/R_t
\]

(9-1)

\[
\beta E[C_tS_{t+1}/C_t] = S_t/R^*_t
\]

(9-2)

\[
E[(S_{t+1} - F_t)/C_{t+1}] = 0
\]

(9-3)

Note that, because 1/C_{t+1} is a convex function of C_{t+1}, it follows that C_t/C_{t+1} < C_tE[1/C_{t+1}] = 1/\beta R_t. It then follows immediately that:

\[
E[C_{t+1}] > C_{t+1}
\]

Thus, it was shown that the optimal consumption level is on average higher
under uncertainty than under certainty. This is consistent with similar results obtained in the economic analysis under uncertainty (e.g., McKenna [1986], chapter 5).

Next from (9) the following relationship, stating that the ratio of the forward to the spot exchange rate is equal to the ratio of the real rate of return of the bonds between the two countries, is derived:

\[ \frac{F_t}{S_t} = R_t / R_t^* \]  

(10)

Subtracting unity from both sides of (10) yields the well-known approximating relationship of the covered interest rate parity (CIP), i.e., Frenkel and Razin [1980].

\[ \frac{(F_t - S_t)}{S_t} = \frac{r_t - r_t^*}{S_t} \]  

(10')

This implies, as is well-known, absence of arbitrage.

Furthermore, solving (9-3) for the forward foreign exchange rate \( F_t \) yields:

\[ F_t = E[S_{t+1}/C_{t+1}] / E[1/C_{t+1}] \]  

(11)

\( E[1/C_{t+1}] \) is nothing but the expected marginal utility of \( U \) with respect to the future consumption level \( C_{t+1} \). Thus, equation (11) means that the relationship between the forward and the future spot exchange rates will be affected by the expected marginal utility of the future real consumption level. Our assumption 1' with respect to a parametrized utility (felicity) function implies that from equation (7-1'), \( C_{t+1} = (1 - \alpha \beta - \gamma)Y_{t+1} \). Thus, under the optimal conditions, (7-2), together with (4), yields:

\[ C_{t+1} = (1 - \alpha \beta - \gamma)\beta K_{t+1}^{\alpha} \gamma Z_{t}^{\alpha} Z_{t+1} \]

Finally, substituting this into (11) yields:

\[ F_t = E[S_{t+1}] + \text{cov}(Z_{t}^{\alpha} Z_{t+1}^{-1}, S_{t+1}) / E[Z_{t}^{\alpha} Z_{t+1}^{-1}] \]  

(12)

Thus, it is proved that the unbiasedness hypothesis does not in general hold

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4. From equation (9-1) \( E[1/C_{t+1}] = 1/(\beta R_t) \), and from equation (9-2) \( E[S_{t+1}/C_{t+1}] = S_t/(\beta R_t^t C_t) \). Substituting these relationships into \( E[S_{t+1}/C_{t+1}] = F_t E[1/C_{t+1}] \) (i.e., equation (9-3)) yields (10).
within a production economy. Statistically, the probability of the hypothesis holding is actually zero. A proposition stating that the unbiasedness hypothesis is constantly disturbed has been well-known both theoretically and empirically.\(^5\) For example, Frenkel and Razin [1980], Engel [1984], and Andersen and Sorensen [1994] pointed out that, when the future prices are kept constant, the agent’s attitudes towards risk, and the initial holding of assets, are responsible for the failure of the unbiasedness hypothesis within an exchange economy. According to Taylor [1995], if the risk-neutral efficient market hypothesis holds, then the unbiasedness of the forward exchange rate necessarily follows, given covered interest rate parity, \((10')\). However, in our production economy, it is interesting to observe in \((12)\) that a more basic technological uncertainty is shown to hinder the realization of the unbiasedness hypothesis.\(^6\)

If the hypothesis holds, \(F_t\) must be equal to \(E[S_{t+1}]\) in equation \((12)\). Also, the CIP is shown to hold in equation \((10')\). On the other hand, if the representative agent is risk averse, then:

\[
\frac{(E[S_{t+1}] - S_t)}{S_t} = \lambda_t = \lambda_t = r_t - r^*_t
\]

must be assumed for the existence of the risk premium, \(\lambda_t\). Therefore, the well-known fact is confirmed here, that is, in a world inhabited by risk-averse individuals, unbiasedness and absence of arbitrage are mutually exclusive. In other words, in general the covariance term in equation \((12)\) must be non-zero. The reasons for it do not lie in those offered by Frenkel and Razin [1980], Engel [1984], Andersen and Sorensen [1994] or Taylor [1995], but in those generated from uncertainty in production.

Finally, characteristics of \(F_t\) and \(R_t\) are examined. First of all, for \(F_t\), equations \((10)\) and \((9-1)\) yield:

\[
F_t = \frac{S_t R_t}{R^*_t} = \frac{S_t E[C_{t+1}/\beta C_t]}{E[C^*_t/\beta C^*_t]}
\]

\(^5\) See MacDonald and Taylor [1992] and Taylor [1995] for references to the existing relevant literature on both empirical and theoretical analyses of the unbiasedness hypothesis of the forward exchange rate.

\(^6\) Unfortunately, the sign of the covariance term in equation \((12)\) cannot be determined at this stage of specification of our model. It should be stressed, however, that a valid reason for assumption 6 lies in this indeterminacy.
If we further assume that the same production condition applies to the foreign country, so that assumptions 2 to 4 are valid, and the foreign variables are identified with asterisks, then substitution into the above relationship yields:

\[ F_t = S_tE[Z_{t+1}Y_{t+1}^*]/E[Z_{t+1}Y_{t+1}^*] \] (13)

Equation (13) was derived under an additional assumption that the production function, together with its parameter, is the same between both countries. Part (b) of assumption 4 is here specified one more step further, and \( Z_{t+1} \) is assumed as follows:

**Assumption 4' Negative autocorrelation**

(b') \( \ln Z_t \) obeys a negative autoregressive scheme, \( 0 > \rho > -1 \).

In other words, this assumption 4 (b') means that an increase in \( Y_t \), ceteris paribus, will decrease \( Y_{t+1} \) through a decrease in \( Z_{t+1} \). Then, using equation (4) and the corresponding foreign country's production function, equation (13) can be rewritten as follows:

\[ F_t = S_tV_tE[Z_{t+2}Z_{t+1}^*/E[Z_{t+2}Z_{t+1}^*]] \]

\[ = S_tV_t\exp\{(1+(\rho+\alpha-1)^2)\sigma_{\alpha}^2/2\} \] (14)

where \( V_t = E[(K_t)^{\alpha-1}]/E[(K_t^*)^{\alpha-1}] > 0 \). The equilibrium relationship (14) contains the following important implications: (a) an increase in \( Y_t \) (\( Y_t^* \)) will decrease (increase) \( F_t \), (b) an increase in \( \sigma_{\alpha}^2 \) will increase (decrease) \( F_t \), (c) an increase (decrease) in \( S_t \) will increase (decrease) \( F_t \), but (d) the effect of a change in \( \alpha \) will depend on the relative size and scale of the home to the foreign country. Furthermore, (e) an increase in \( \rho \) will increase \( F_t \), because its increase in time \( t-1 \) will decrease \( Z_t \) and, hence \( Y_t \). This decrease in \( Y_t \) will make \( F_t \) depreciated.

Next, the relationship \( R_t = E[C_{t+1}/\beta C_t] = 1+r_t \) which is transformed from equation (9-1) is examined. This relationship is further rewritten as:

\[ R_t = (1/\beta)E[(\alpha \beta)^\alpha Z_{t+1}Y_{t+1}^*/\alpha^\alpha \beta^\alpha E[Z_{t+1}Y_{t+1}^*]] \] (15)

Applying the previous additional assumption 4' to (15), it is further simplified as:
\[ R_t = \alpha^\beta a^{-1} \mathbb{E} \left[ K_t a^{(\alpha-1)}Z_{t+1}a^{-1} \right] \]
\[ = \alpha^\beta a^{-1}K_t a^{(\alpha-1)}Z_{t+1}a^{(\rho+a-1)} \exp \left[ 1 + (\rho + \alpha - 1)^2 \right] \sigma_w^2 / 2 \]  

The following implications are deduced from the equilibrium relationship (16). That is, (a) the well-known countercyclical movement is observed on average between \( Y_t \) and \( R_t \) (and hence \( r_t \)) [e.g., Salyer [1994]], (b) an increase in the rate of time preference \( \beta \) unambiguously decreases \( r_t \) (and hence \( R_t \)), while the effect of a change in the degree of concavity \( \alpha \) in the production function is ambiguous, (c) an increase in technological uncertainty \( \sigma_w^2 \) will increase \( r_t \) (and hence \( R_t \)), and (d) an increase in \( K_t \) will decrease \( r_t \) (and hence \( R_t \)).

Intuitively, (a) means that an increase in \( Y_t \) brings about, through the optimal life-cycle consumption decision, an increase in the demand for bonds, which in turn leads to a decrease in the rate of interest. The implication (b) shows that, because an increase in the rate of time preference can enhance the current felicity level with less current consumption, the demand for bonds also increases. As has already been pointed out, in general, the optimal consumption level under uncertainty is larger than that under certainty. Corresponding to this fact, (c) means that the optimal saving is smaller under uncertainty than under certainty, so that a decrease in the demand for bonds gives rise to an increase in the rate of interest. The relationship (d) will not need further explanation.

IV. Conclusions

The purpose of this article rests on the consistent study of the equilibrium analysis. By assuming a standard real business cycle model, the equilibrium analysis, which has been considered using different models in the traditional monetary theory or economics of uncertainty, was conducted within a model of a simple production economy. Our analysis was also partly motivated by the intrinsic deficiency inherent to economic models of price uncertainty which has been criticized by Kemp [1976] and Tumovsky [1976]. This important problem was partially resolved by assuming production uncertainty in an open economy. In order to perform the exercise, the optimal consumption decision, together with the simultaneous investment
(and hence output) decision, was first examined within a growth model with stochastic technological shocks. At the next stage of the optimal portfolio decision analysis, the relationship between the forward exchange rate and the expected future spot exchange rate was considered. At first glance, the relationship seemed to be similar to the one obtained within a pure exchange economy model, but the analysis clearly showed that a completely different element of technological uncertainty actually plays an important role in the relationship between the forward exchange rate and the expected future spot exchange rate. In the last stage, some equilibrium characteristics between the forward exchange rate and the interest rate, which were derived from the assumed model, were closely analyzed. All of them were convincing because they are intuitively appealing to common sense. It should be emphasized once more that those important characteristics were made explicit here within a simple production economy.

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