Journal of Economic Integration
7(2), Autumn 1992, 137-150

The Dynamic Rybczynski Theorem and Its Dual

Ngo Van Long*
Australian National University

Abstract

In the context of capital accumulation, it is shown that there exists a dynamic version of the Rybczynski Theorem, and of the Stolper-Samuelson Theorem. If, in addition, the investment good is labour intensive, then a dynamic reciprocity relation is also obtained. In the case where the investment good is capital intensive, only a weak form of duality between the two theorems can be established. The paper makes use of the dynamic envelope results of Caputo, and of Lafrance and Barney.

I. Introduction

A central result in trade theory is the Rybczynski Theorem, which states that if an economy produces two goods using two factors of production (say capital and labour), under neoclassical technology with constant returns to scale, then an increase in its endowment of capital will result in an expansion of the output of the capital intensive good and a contraction of the output of the labour intensive good, provided that the price ratio is exogenously given. The dual of this theorem is the Stolper-Samuelson Theorem, which says an increase in the price of the capital intensive good will result in an increase in the rental rate and a decrease in the wage rate. The duality is most striking since for marginal changes the magnitude of the Rybczynski effect is exactly equal to that of the Stolper-Samuelson effect:

$$\partial X / \partial K = \partial r / \partial p$$

* Department of Economics, Faculty of Economics & Commerce, GPO Box 4, Canberra, ACT 2601, Australia. I would like to thank Murray C. Kemp, Alan Woodland, and a referee for helpful comments.

where r is the rental rate, K is the economy's stock of capital, X is the output of the capital intensive good, and p is its price. The above equation was one of Samuelson's famous *reciprocity relations*. (See Samuelson [1953-54].)

The purpose of this paper is to determined the extent to which the duality between the Rybczynski effect and the Stolper-Samuelson effect can be generalised to dynamic models. The idea of a dynamic Rybczynski Theorem has been explored by Kemp and Long [1979, 1982, 1984] in the context of an economy using an exhaustible resource and labour to produce two goods. The present paper does not deal with exhaustible resources; if focuses instead on the dynamics of capital accumulation, and on the duality results. Its main findings are that there exists a dynamic version of the Rybczynski Theorem, and of the Stolper-Samuelson Theorem, that a strong form of duality obtains if the investment good is labour intensive (a dynamic reciprocity relation can be proved in this case), and that in the case where the investment good is capital intensive then only a weak form of duality exists. The dynamics of trade and capital accumulation has been carefully studied by Oniki and Uzawa [1965], and Manning [1980]. However neither paper addressed the issues raised in this paper.

This paper makes use of the dynamic envelope results of Caputo [1990] and LaFrance and Barney [1991], as well as some useful properties of value functions, as expounded in Leonard and Long [1992].

II. The Model

Consider a small open economy producing two goods using capital and labour. The labour force is constant and is normalised at unity. Let X and Y denote respectively the output of the capital intensive good and that of the labour intensive good. Let p be the international price of X in terms of Y. It is assumed that p is time-invariant.[1]

Assume for the moment that the consumption good is capital intensive, and the capital good is labour intensive. This is in keeping with the standard descriptive

1. This assumption is made to simplify the analysis. A more general parametrization of the price path would not affect the essential results. In fact Caputo [1992] has shown that the qualitative properties of comparative dynamic results with time independent parameters are identical to those with time dependent parameters, provided that each of these parameters is perturbed independently by a parallel displacement.

growth models (see Burmeister and Dobell [1970], for example). Let K denote the country's stock of capital, and I the rate of investment. Assume for simplicity that there is no depreciation.[2] Then

$$dK / dt = I \tag{1}$$

Total capital expenditure at any time consists of I and installation costs, denoted by $g(I)$. We take it that $g(I)$ is a strictly convex function, and

$$g(0) = 0,\ g(I) > 0 \text{ for } I \neq 0,\ g'(0) = 0. \tag{2}$$

The world rate of interest is i, which we suppose to be time-invariant and exogenously given to the home country. The intertemporal budget constraint for our small open economy is

$$\int_0^\infty \exp(-it)\{pC + I + g(I)\}dt = \int_0^\infty \exp(-it)\{pX + Y\}dt \tag{3}$$

where C denotes the consumption expenditure. The left-hand side of equation (3) is the present value of the streams of consumption and investment expenditures and the right-hand side is the present value of the stream of gross domestic product. It is convenient to re-write (3) as

$$\int_0^\infty \exp(-it)pCdt = \int_0^\infty \exp(-it)\{pX + Y - I - g(I)\}dt \tag{4}$$

Let $<C(t)>$ denote an entire time path of consumption. We assume that the country wants to maximize a welfare functional $U\{<C(t)>\}$, which is increasing in C. Since the international price path and the interest rate are both exogenously given, production and consumption decisions are separable, i.e. the Fisher separation theorem applies. It follows that we can study the time paths of production and capital accumulation without specifying the utility functional. We focus therefore on the problem of maximizing the present value of the stream of net output, subject to the initial condition

$$K(0) = K_0 \tag{5}$$

---

2. Adding depreciation would not change the result. Of course net rental (gross rental minus depreciation) rather than gross rental would be equated to the interest rate in a steady state.

and the production possibility function

$$G(X,Y,K)=0,\ X\geq 0,\ Y\geq 0, \tag{6}$$

where labour has been omitted, since the labour force is a constant.

Our problem can therefore be stated as that of finding (X, Y, I) that solve

$$V(p,K_0)=\max_{X,Y,I}\int_0^{\infty}\exp(-it)\{pX+Y-I-g(I)\}dt \tag{7}$$

subject to (1), (5) and (6).

Since X and Y do not appear in the transition equation (1), and I does not appear in the constraint (6), we can solve problem (7) in two steps. First, for given K, we choose X and Y to maximize $pX+Y$ subject to (6), thus obtaining the National Income function $N(p,K)$. In the second step, we substitute $N(p,K)$ into (7) and then optimize with respect to I.

For given K and p, the National Income function (see Woodland [1982, pp.58-59]) is defined as follows:

$$N(p,K)=\max_{X,Y} pX+Y \tag{8}$$

subject to

$$G(X,Y,K)=0,\ X\geq 0,\ Y\geq 0.$$

Let $F_1(K)$ [respectively $F_2(K)$] be the output of X [respectively Y] if the country completely specializes in the production of that good. Applying the envelope theorem to (8), it is seen that the function $N(p,K)$ has the following properties:

$$\partial N/\partial p=X(p,K) \tag{9}$$

$$\partial N/\partial K=\lambda(p,K)G_K(X(p,K),Y(p,K),K)\equiv r(p,K) \tag{10}$$

where r is interpreted as the rental rate in terms of the numeraire good and λ is the optimal value of the Lagrange multiplier for problem (8). It is known that there exist two critical values $\underline{K}(p)$ and $\overline{K}(p)$ such that

$$N(p,K)=pX=pF_1(K)\quad \text{if } K>\overline{K}(p) \tag{11}$$

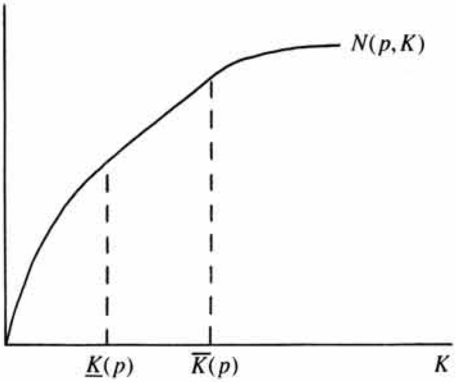

Figure 1

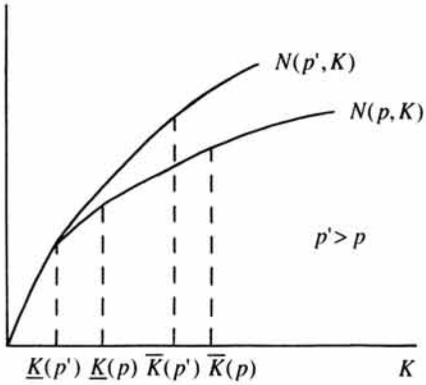

Figure 2

$$N(p,K) = Y = F_2(K) \qquad \text{if } K < \underline{K}(p) \tag{12}$$

$$N(p,K) = pL_1F_1[\overline{K}(p)] + (1-L_1)F_2[\underline{K}(p)] \text{ if } \underline{K}(p) \le K \le \overline{K}(p) \tag{13}$$

where L_1 is defined by

$$L_1\overline{K}(p) + (1-L_1)\underline{K}(p) = K. \tag{14}$$

The function *N(p, K)* is illustrated in Figure 1. For *K* in the interval $[\underline{K}, \overline{K}]$, the function is linear, with slope S*, where

$$S^* = pF_1'[\overline{K}(p)] = F_2'[\underline{K}(p)]. \tag{15}$$

The critical values $\underline{K}$ and $\overline{K}$ are both decreasing functions of *p*, as illustrated in Figure 2. These properties are useful for our derivation of the dynamic Rybczynski theorem.

For later reference, we state the following lemma.

Lemma 1**:** $N_{pK} \ge 0$, *with strict inequality of X is positive.*

Proof**:** From (9),

$$N_{pK} = \partial X / \partial K. \tag{16}$$

Since good X is capital intensive, the static Rybczynski theorem implies that if X is positive an increase in K will lead to an increase in X. If X is initially zero, then a marginal increase in K has no effect on X. This complete the proof.

In that follows, in order to ensure that in a steady state the capital stock is positive and finite, we impose the conditions

$$F_2'(0) > i \tag{17}$$

$$pF_1'(\infty) < i. \tag{18}$$

We can now redefine the value function $V(p, K_0)$ as follows

$$V(p, K_0) = \underset{I}{Max} \int_0^\infty \exp(-it)\{N(p, K) - I - g(I)\}dt \tag{19}$$

subject to

$$dK / dt = I \tag{20}$$

and

$$K(0) = K_0 \tag{21}$$

The current-value Hamiltonian associated with problem (19) is

$$H = N(p, K) - I - g(I) + qI \tag{22}$$

where q is the co-state variable, or the current value shadow price of capital. The necessary conditions are (20), (21) and

$$q - 1 - g'(I) = 0 \tag{23}$$

$$dq / dt = iq - \partial N / \partial K \tag{24}$$

Since our assumptions (17) and (18) ensure that a steady state K^s exists, we can state the boundary condition

$$\lim_{t \to \infty} K(t) = K^s. \tag{25}$$

It is clear that (25) implies that for all finite $q(t)$,

$$\lim_{t \to \infty} q(t)K(t)\exp(-it) = 0 \tag{26}$$

Using the usual phase diagram analysis, it can be shown that if $F_2'(\underline{K}(p)) < i$, then the steady state capital stock is K_2, defined by

$$F_2'(K_2) = i \tag{27}$$

and at the steady state, the country specializes in the labour intensive good. In the case where $pF_1'(\overline{K}(p)) > i$, the steady state capital stock is K_1, defined by

$$pF_1'(K_1) = i \tag{28}$$

and the steady state production is specialized in the capital intensive good. Finally, in the remaining case, with

$$F_2'(\underline{K}(p)) = i = pF_1'(\overline{K}(p)) \tag{29}$$

there is a continuum of steady states. Figures 3, 4 and 5 are the phase diagrams for the three cases. In Figures 3 and 4 the unique steady state has the saddlepoint

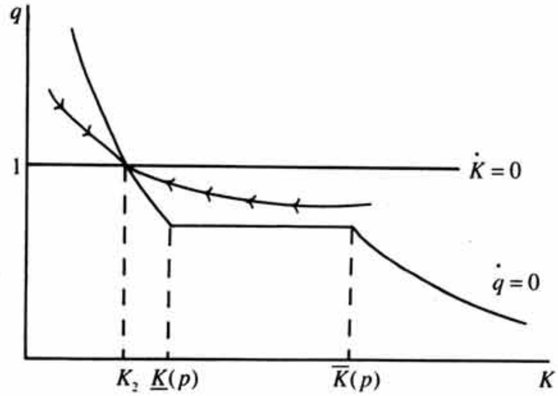

Figure 3

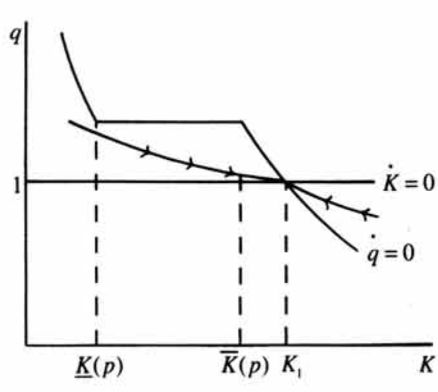

Figure 4

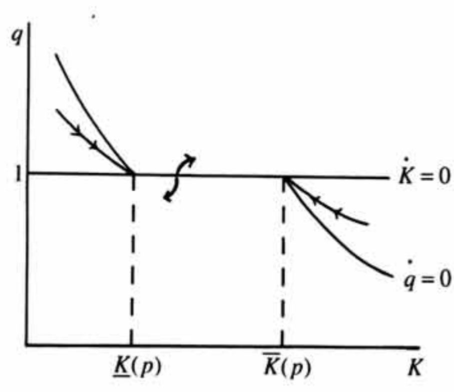

Figure 5

property, and starting at any initial capital stock $K_0 > 0$, the system converges to the steady state. In Figure 5, if $K_0 < \underline{K}(p)$ then capital will be accumulated and $\underline{K}(p)$ is approached asymptotically; if $K_0 > \overline{K}(p)$ then the capital stock will be run down toward $\overline{K}(p)$; and if $\overline{K}(p) \geq K_0 \geq \underline{K}(p)$, then the system will remain there.

We are now ready to prove a dynamic version of the Rybczynski theorem.

III. A Dynamic Rybczynski Theorem

The static version of the Rybczynski Theorem assumes incomplete specialisation. In view of equation (29) and the ensuing discussion, if a country starts at a steady state with diversification, a small increase in its capital stock will immediately bring the country to a new steady state, adjacent to the old one, with complete diversification and the results of the static Rybczynski Theorem apply without modification. We now consider more interesting cases.

Suppose that $F_2'(\underline{K}(p)) < i$, so that the steady state production is completely specialized in the labour intensive good. Let $K_0 > \underline{K}(p)$, so that production is diversified over some time interval along the optimal path to the steady state. In this case an increase in K_0 will result in an increase in the output of the capital intensive good at each point in time when it is produced, and a decrease in the output of the labour intensive good at each point in time. To prove this, we first note that if the initial stock is $K_{00} > K_0$, then the optimal path is $K^{**}(t) > K^{*}(t)$ for all t, except at the steady state. The static Rybczynski theorem, referred to in the introduction and in the proof of Lemma 1, then implies that at all points of time where both goods are produced, the output of X will be higher and, that of Y lower, along the path $K^{**}(t)$, as compared with path $K^{*}(t)$.

Consider next the case where $pF_1'(\overline{K}(p)) > i$, so the steady state production is completely specialized in the capital intensive good. Let $K_0 < \overline{K}(p)$, so that along the optimal path there is a phase with diversification. A similar argument applies, and one can see that at each point of time where X is positive, more of it is produced along $K^{**}(t)$, as compared with along $K^{*}(t)$. If $K_0 < \underline{K}(p)$ then it is not true that less of the labour intensive good is produced at each point of time.

We can now state our first proposition.

Proposition 1: *(A dynamic Rybczynski Theorem): An increase in the initial endowment of capital will expand the output of the capital intensive good at each point of* *time when it is produced; the output of the labour intensive good will be reduced at each point of time, provided that the initial capital stock is not in the range of complete specialization in that good.*

Let us now define *The Rybczynski Effect* as the response of the cumulative discounted output of the capital intensive good to an increase in the initial endowment of capital. Formally, let

$$X_c = \int_0^\infty \exp(-it)X(t)dt \tag{30}$$

The Rybczynski effect is defined as

$$R = \partial X_C / \partial K_0 > 0 \tag{31}$$

where R is positive by virtue of Proposition 1.

Consider the value function $V(p, K_0)$ defined by (19). Using the dynamic envelope theorem of Caputo, LaFrance and Barney, we can calculate the partial derivative of V with respect to p:

$$\partial V / \partial p = \int_0^\infty \exp(-it)(\partial N / \partial p)dt \tag{32}$$

Combining (32) and (9):

$$\partial V / \partial p = \int_0^\infty \exp(-it)X(t)dt = X_C \tag{33}$$

From (31) and (33), the Rybczynski effect is the second cross partial derivative of the value function:

$$R = V_{pK_0} \tag{34}$$

The result will prove useful in establishing the duality between the Rybczynski Theorem and the Stolper-Samuelson Theorem.

It should be noted that one can prove the positivity of R directly from (34):

$$V_{pK_0} = \int_0^\infty \exp(-it)[\partial X / \partial K^*(t)]\,[\partial K^*(t) / \partial K_0]dt \tag{35}$$

where $\partial X / \partial K^*(t)$ is positive for $K^*(t) > \underline{K}(p)$ and zero for $K^*(t) \le \underline{K}(p)$, and $\partial K^*(t) / \partial K_0$ is positive.

The result conveyed by (34) and (35) may be regarded as a corollary to Proposition 1:

***Corollary 1**: An increase in the initial capital stock leads to an increase in the cumulated discounted output of the capital intensive good.*

IV. A Dynamic Stolper-Samuelson Theorem

Consider now the effect of an increase in the price of the capital intensive good. First note that from (10), (23), (24) and $g'(0)=0$, in a steady state the real rental rate must equal the rate of interest *i*. We also know that if a country produces both goods at some time *t*, then the rental rate at that time is uniquely determined by the intersection of the two isocost curves $p=C_1(w,r)$ and $1=C_2(w,r)$. It follows that given the rate of interest *i*, there exists for each small open economy *j* a unique relative price *p*(*j*) which gives rise to a steady state with diversification for that country. (We do not assume that all countries have the same production functions.) Given the interest rate *i*, if a country diversifies in a steady state, a change in *p* will create a new steady state for that country, with specialization.

It follows from the observations made in the preceding paragraph and from Lemma 1 and other properties of the National Income function $N(p,K)$ that starting from any steady state with a positive output of good *X*, an increase in its price *p* will lead to a larger steady state capital stock. The steady state rental rate remains constant, however. The proofs of these results in the case of complete specialisation are instructive. Note that in the steady state $q^s=1$, so $K=K^s(p,i)$ must satisfy

$$N_k(p,K)-i=0$$

Applying the implicit function theorem to the above equation, we have

$$\partial K^s / \partial p = -N_{Kp} / N_{KK}$$

With $N_{KK}<0$ and $N_{KP}>0$ by Lemma 1, the above derivative is positive. As for the rental rate, we have in the steady state

$$r^s(p,i)=N_k(p,K^s(p,i))$$

and thus the derivative of r^s with respect to p is

$$\partial r^s / \partial p = N_{Kp} + N_{KK}(\partial K^s / \partial p) = N_{Kp} - N_{KK}(N_{Kp} / N_{KK}) = 0$$

The above equation resembles a Slutsky-like expression.[3] The first term is the short run impact of the price change on the rental rate, holding the capital stock fixed, and the second term is the indirect effect of p on r by allowing the capital stock to adjust. The sum of these effects is zero. What about the discounted stream of rental earned by a unit of capital?
From (10) and (24):

$$dq / dt - iq = -r \tag{36}$$

Integrating (36), and using the transversality condition (25), we have

$$q(0) = \int_0^\infty \exp(-it) r(t) dt \tag{37}$$

The right-hand side of (37) is the present value of the stream of rental earned by a unit of capital. Therefore we can define the Stolper-Samuelson effect as:

$$S = \partial q(0) / \partial p \tag{38}$$

We know from the properties of value functions of optimal control problems that

$$q(0) = \partial V / \partial K_0 \tag{39}$$

It follows that

$$S = V_{K_0 p} = V_{p K_0} = R > 0 \tag{40}$$

We can thus state our second proposition:
Proposition 2: (*A dynamic Stolper-Samuelson Theorem*): *An increase in the price of the capital intensive good will increase the present value of the stream of rental earned by a unit of capital. In addition the following reciprocity relation holds*

$$S = R. \tag{41}$$

3. I am indebted to a referee for pointing this out, and for offering the above proof.

V. The Case of Capital Intensive Investment Goods

So far we have assumed that the investment good is labour intensive. We now turn to the opposite case. Equations (3), (4), (7), (19) and (22) must be modified: instead of $I + g(I)$ we now have $pI + g(I)$. Equation (23) becomes

$$q - p - g'(I) = 0 \tag{43}$$

and in a steady state we have $r / p = i$. Equations (32) – (35) are no longer valid. They must be replaced by

$$\partial V / \partial p = \int_0^\infty \exp(-it)\{(\partial N / \partial p) - I\} dt \tag{32'}$$

$$\partial V / \partial p = \int_0^\infty \exp(-it)\{X - I\} dt \tag{33'}$$

$$R = V_{pK_0} + \int_0^\infty \exp(-it)\ [(\partial I / \partial K^*]\ [\partial K^* / \partial K_0]\ dt \tag{34'}$$

$$V_{pK_0} = \int_0^\infty \exp(-it)\ \{(\partial X / \partial K^*) - (\partial I / \partial K^*)\}[\partial K^* / \partial K_0]\ dt \tag{35'}$$

It remains true that R is positive, since Proposition 1 was proved without using (32) – (35). The integral expression in (35') is negative because

$$\partial I(t) / \partial K^*(t) = \{\partial I(t) / \partial q^*(t)\}\{\partial q^*(t) / \partial K^*(t)\} \tag{43}$$

and $\partial I(t) / \partial q^*(t)$ is negative from (23), and $\partial q^*(t) / \partial K^*(t)$ is non-positive because the stable branch of the saddlepoint is downward sloping.

Since the Stolper-Samuelson effect, S, is equal to $V_{K_0 p}$ we have

$$S > R > 0 \tag{44}$$

To summarize, if the investment good is capital intensive then Propositions 1 and 2 remain valid, except the reciprocity relation (41) no longer holds and must be replaced by the inequality (44).

IV. Conclusion

We have been able to obtain a dynamic version of the Rybczynski Theorem and of the Stolper-Samuelson Theorem. We also showed that the reciprocity relation holds if the investment good is labour intensive.

Within the static framework, both theorems have been generalized to the case of many goods and factors. (See Chang, Ethier, and Kemp [1980], Kemp and Wegge [1969], Ethier [1984], Jones *et. al.* [1990].) The possibility of extending these generalizations to dynamic models remains an open question.

References

Burmeister, E. and R. Dobell [1970], *Mathematical Theories of Economic Growth*, Macmillan, London.

Caputo, M.R. [1990], "Comparative Dynamics via Envelope Methods in Variational Calculus," *Review of Economic Studies*, Vol. 57, pp. 689-97.

Chang, W., W. Ethier and M.C. Kemp [1980], "The Theorems of International Trade with Joint Production," *Journal of International Economics*, Vol. 10, pp. 377-94.

Ethier, W. [1984], "Higher Dimensional Issues in Trade Theory," in R.W. Jones and R. Kenen (eds.), *Handbook of International Economics*, Vol. I, Elsevier, Amsterdam.

Jones, R.W. [1965], "Duality in International Trade: A Geometrical Note," *Canadian Journal of Economics and Political Science*, Vol. 31, pp. 390-93.

Jones, R.W., S. Marjit and T. Mitra [1990], "The Stolper-Samuelson Theorem: Links to Dominant Diagonals," Working Paper No. 221, Rochester Center for Economic Research.

Kemp, M.C. and N.V. Long [1979], "International Trade with an Exhaustible Resource: A Theorem of Rybczynski Type," *International Economic Review*, Vol. 20, pp. 671-677.

Kemp, M.C. and N.V. Long [1982], "Rybczynski Theorem in a Context of Exhaustible Resources: The Case of Time-Contingent Prices," *International Economic Review*, Vol. 23, pp. 699-710.

Kemp, M.C. and N.V. Long [1984], "The Role of Natural Resources in Trade Models," in R.W. Jones and P. Kenen (eds.), *Handbook of International Economics*, Vol. I, Elsevier, Amsterdam.

Kemp, M.C. and L. Wegge [1969], "On the Relation Between Commodity Prices and Factor Rewards," *International Economic Review*, Vol. 10, pp. 407-413.

LaFrance, J.T. and L.D. Barney [1991], "The Envelope Theorem in Dynamic Optimization," *Journal of Economic Dynamics and Control*, Vol. 15, pp. 355-385.

Leonard, D. and N.V. Long [1992], *Optimal Control Theory and Static Optimization in Economics*, Cambridge University Press, New York.

Manning, R. [1981], "Specialization and Dynamics in a Trade Model," *Economic Record*, Vol. 57, pp. 251-260.

Oniki, H. and H. Uzawa [1965], "Patterns of Trade and Investment in a Dynamic World of International Trade," *Review of Economic Studies*, Vol. 32, pp. 15-38.

Samuelson, P.A. [1953-54] "Prices of Factors and Goods in General Equilibrium," *Review of Economic Studies*, Vol. 21, pp. 1-20.

Woodland, A.D. [1982], *International Trade and Resource Allocation*, North-Holland, Amsterdam.